Speculative Bubble Burst

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Abstract: Central to market fundamentals are three ideas: (1) Nominal money (2) Dividend (3) Existing stock. In connection with the cumulative dividend stream criterion of fundamental and noise movement, the conception of sequentially stable Markov process is grounded on the theory of bubbles. This paper firstly embodies the origin of speculative bubble burst with overconfidence. Then, unique equilibrium with inertia is re-illuminated by the overconfidence.

Keywords: externalities, speculative bubbles, heterogeneous beliefs, overconfidence, speculative bubble burst, equilibrium with inertia.

JEL: D01; D52; D62; D84

1. Introduction

By applying the equilibrium-conceptual approach to stability of markets, economic and financial crisis from 2008 illuminates familiar problems from a new angle, to pose provocative questions that the price level changes rapidly even though economic *fundamentals* are seemed to have no problem. Neoclassicals, mainstream scholars in the Microeconomic foundation, ferrot out this key problem by the axiomatic approach. In Microeconomics, classically, *economic equilibrium* is a state where economic forces such as supply and demand are balanced without externalities and equilibrium values of variables are stabilized. In what follows, we explore the case with *externalities* with (1) shocks (2) speculative bubbles and (3) an inertia mechanism.

We offer a new interpretation of *overconfidence* as motivation not to participate in economic activities immediately. Risk-averse agents provide a much more rational way to solve the puzzle of externalities. If beliefs of traders are *concordant* (Milgrom and Stokey, 1982): traders agree about how information should be interpreted, Concordant beliefs arise naturally in statistical problems where there is an unknown parameter about which traders may hold different views.

In connection with this belief, the basic framework of trading can take the financing source as a focal point:

(1) Fiat money (Santos and Woodford, 1997).
(2) Self-enforcing debt (Hellwig and Lorenzoni, 2009).

Let us initiate a discussion of the separation between asset pricing bubbles and speculative bubbles. For example, during the German hyperinflation, Flood and Garber (1980) had immense appeal for hypothesis of no speculative bubbles. However, theoretically, financial market setting emphasizes on finite wealth, the finite horizon and short-selling behavior of risk-averse finitely lived agents. Hence, the bubble condition cannot be paralleled with transversality or boundary conditions.
The methodological part of speculative bubble has come to debate on a *Markov process* and an *equilibrium.* The article "Sunspots and Cycles" (Azariadis and Guesnerie, 1986) is devoted to the detailed technique with sequentially stable Markov process in accordance with perfect foresight. Unstable beliefs in the form of the transition probability matrix are intended to yield a regular stationary sunspot equilibrium (SSE).
In this article, market fundamentals becomes apparent with three concepts of O.Blanchard (1979) in the section 2. The section 3 introduces the infinitesimal generator to cumulative dividend process. In the section 4, psychological factors of agents like aggregated beliefs, overconfidence carry implications for a speculative bubble crush with an infinitesimal generator. In the section 5, origin of speculative bubble burst and unique trading point are addressed. In the section 6, existence of unique equilibrium allocation with inertia at the speculative bubble crush is described. I shall proceed in the following with regard to the literature.

2. Axiomatic Foundations of Speculative Bubbles

Market fundamentals rank the price level x_t in a discrete-time framework according to

$$x_t = f_t + aE(x_{t+1} \mid \Omega_t), a \in (0, 1), t \in \mathbb{N}$$

where x_t is the price level, f_t is fundamentals and Ω_t is the information set.

It becomes apparent with three examples suggested by O. Blanchard (1979):

1. Reduced form of a money market equilibrium: x_t is price level and f_t is nominal money (Flood and Garber, 1980)

2. Arbitrage equation: x_t is price of share and f_t is dividend (Scheinkman and Xiong, 2003)

3. Equilibrium of a material market (for example the gold market): x_t is price and f_t is existing stock (Harrison and Kreps, 1978)

No single answer about the definition of bubbles will suffice but it is highly probable that *bubbles* are price deviations from their fundamentals similar to Allen and Gorton (1993). Still, it bristles with ambiguity how price simply goes down when bubble bursts even though bubble is itself a rich assembly of determinants. Scheinkman and Xiong (2003) formulated a criterion linked to the continuous-time cumulative dividend stream. The buyer's willingness to pay is a function of the value of the option that he acquires, the payoff from stopping is (Stokey, Lucas and Prescott, 1989), in turn, related to the value of option like American options with no maturity. The continuous-time cumulative dividend process D_t is given by

$$dD_t = f_t dt + \sigma_D dZ_t^D, \tag{1}$$

where f is the fundamental random variable, σ_D is a constant volatility parameter and Z_t^D denotes the standard Brownian motion of the process D_t. The value of process f_t is defined by

$$df_t = -\lambda(f_t - \overline{f})dt + \sigma_f dZ_t^f, \tag{2}$$

where mean reversion is non negative such that $\lambda \geq 0$, and $\overline{f}$ denotes long-run mean of f.

The idea centered on the speculative behavior if the right to resell stock makes them willing to pay more. Relevant to "churn bubbles" of Allen and Gorton (1993), fund managers are ready to negotiate trading above their fundamental even though bubble can crash by a bad luck.

3. Dividend Process and the Infinitesimal Generator "A" of the parameter T_t

Let us induce the fundamental and noise movement from the dividend stream in equation (3) with an infinitesimal generator "A" defined by

$$Af = \lim_{t \to 0} \frac{(T_t f - f)}{t},$$

where T_t satisfies the property of a feller semigroup. Thus, I define the cumulative dividend process by

$$dD_t = Af_t dt + \sigma_D dZ_t^D, \tag{3}$$

where the value of f is given in equation (2).

It is worth noting that the collection of parameters denoted by $\{T_t\}_{t\geq 0}$ is a feller semigroup ($\equiv T_{t+s} = T_t \circ T_s$). It is positive linear and contracting mapping from the space of all real-valued continuous functions on a locally compact topological space (bounded borel function, Revuz and Yor, 1991) with a countable base that vanish at infinity. The contraction mapping is equipped with the sup-norm $\|\cdot\|$ for all $t \geq 0$ such that

$$\|T_t f\| \leq \|f\|,$$

and satisfies properties like resolvent and spectrum (Revuz and Yor, 1991).

The resolvent of a feller process (or semigroup) is a collection of maps $(A_\lambda)_{\lambda>0}$ for any $\lambda \in R_+$ defined by

$$A_\lambda f = \int_0^\infty e^{-\lambda t} T_t f dt$$

It can be shown that it satisfies the identity.

$$A_\lambda A_\mu = A_\mu A_\lambda = (A_\mu - A_\lambda)/(\lambda - \mu).$$

The spectrum is the complement of the resolvent set: $\sigma(A) = R - A_\lambda$.
If an infinitesimal generator A fails to maintain the uniform limit of fundamental random variable $Af_t dt$, it means that A is not resolvent and no spectrum of A existed in

$$Af = \lim_{t\to 0} \frac{(T_t f - f)}{t}.$$

Fleshing out the infinitesimal generator takes a highly microscopic approach to the question of bubble phenomenon that fundamentals have the tendency to become noise factors during crisis. In examining the speculative bubble crush, I take up memory-less property of Brownian motion.

- A Brownian motion is an example of feller processes.
- Every feller process satisfies the strong Markov property (memory-less).
- A stochastic process has the Markov property if the conditional probability distribution of future state of the process depends only upon the present states defined in terms of a random variable known as a stopping time.

4. Aggregation of Beliefs

Consider two groups of agents $\mathcal{A}$ and $\mathcal{B}$ who observes a vector of their own signals $s^{\mathcal{A}}$ and $s^{\mathcal{B}}$. Assume that heterogeneous beliefs increased by overconfidence offer joint dynamics of the $D, f, s^{\mathcal{A}}, s^{\mathcal{B}}$. All agents observe a vector of each signal s^{A} and s^{B} that satisfy:

$$ds_t^{\mathcal{A}} = A^{\mathcal{A}} f_t dt + \sigma_s dZ_t^{\mathcal{A}},$$

similarly for $ds_t^{\mathcal{B}}$. Here, agents of group $\mathcal{A}(\mathcal{B})$ believe that innovations $dZ^{\mathcal{A}}$ $(dZ^{\mathcal{B}})$ in the signal $s^{\mathcal{A}}(s^{\mathcal{B}})$ are correlated with the common volatility of signals σ_s and the innovation dZ^f in the fundamental process. This process is by Bayesian analysis of the normal distribution.

$$a(dD_t - ds_t^{\mathcal{A}})^2 + b(dD_t - ds_t^{\mathcal{B}})^2 \approx (a+b)\left[x - \frac{(ads_t^{\mathcal{A}} + bds_t^{\mathcal{B}})}{(a+b)}\right]^2 + \frac{ab}{(a+b)(ds_t^{\mathcal{A}} - ds_t^{\mathcal{B}})^2}$$

The factor $\dfrac{(ads_t^{\mathcal{A}} + bds_t^{\mathcal{B}})}{(a+b)}$ has the form of a weighted average of $ds_t^{\mathcal{A}}$ and $ds_t^{\mathcal{B}}$. $\dfrac{ab}{(a+b)} = (a^{-1} + b^{-1})^{-1}$.

Remark 4.1. The factor $(ads_t^{\mathcal{A}} + bds_t^{\mathcal{B}})/(a+b)$ has the form of a weighted average of $ds_t^{\mathcal{A}}$ and $ds_t^{\mathcal{B}}$ along $ab/(a+b) = (a^{-1} + b^{-1})^{-1}$. For combination of a and b, it's necessary to reciprocate, add, and reciprocate the result again to get back into the origin units. $ab/(a+b)$ is one-half of the harmonic mean of a and b.

If agents in group $\mathcal{A}$ trust more their own signal s^{A} than the true information, agents are highly confident to undertake some speculative action. Intentionally, the overconfidence about fundamentals is divided into the part of fundamentals, signal σ_s valued partial fundamentals dZ_t^f correlated with overconfidence ϕ $(0 < \phi < 1)$ and signal σ_s valued noises $dZ_t^{\mathcal{A}}$ which is fluctuated by the residual oscillation of determined correlation with overconfidence. Hence,

$$ds_t^{\mathcal{A}} = A^{\mathcal{A}} f_t dt + \sigma_s \phi dZ_t^f + \sigma_s \sqrt{1-\phi^2} dZ_t^{\mathcal{A}}.$$

and

$$ds_t^{\mathcal{B}} = A^{\mathcal{B}} f_t dt + \sigma_s \phi dZ_t^f + \sigma_s \sqrt{1-\phi^2} dZ_t^{\mathcal{B}}.$$

Regardless of overconfidence = 0, differences of beliefs are according to:

$$g^{\mathcal{A}} = A^{\mathcal{B}} \hat{f}^{\mathcal{B}} - A^{\mathcal{A}} \hat{f}^{\mathcal{A}},$$
$$g^{\mathcal{B}} = A^{\mathcal{A}} \hat{f}^{\mathcal{A}} - A^{\mathcal{B}} \hat{f}^{\mathcal{B}} = -g^{\mathcal{A}}.$$

Whence:

$$
\begin{aligned}
dg^{\mathcal{A}} &= A^{\mathcal{A}} d\hat{f}^{\mathcal{B}} - A^{\mathcal{B}} d\hat{f}^{\mathcal{A}} \\
&= -\rho g^{\mathcal{A}} dt + \sigma_g dW_g^{\mathcal{A}}.
\end{aligned}
$$

Overconfidence is newly coming up after heterogeneous beliefs' movement when it comes to need to distinguish fundamentals and noises depending to ρ and σ_f. Coefficients of Sheinkman and Xiong depend upon parameters of the model $(\lambda, \sigma_D, \sigma_f, \sigma_s, \phi)$ with an infinitesimal generator A according to:

$$
\begin{aligned}
\rho &= \sqrt{A\left(\lambda + \phi\frac{\sigma_f}{\sigma_s}\right)^2 + (1-\phi^2)\sigma_f^2\left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2}\right)}, \\
\sigma_g &= \sqrt{2}\phi\sigma_f
\end{aligned}
$$

The expressions for the second agent are analogously derived as:

$$
\begin{aligned}
dg^{\mathcal{B}} &= dA^{\mathcal{A}} \hat{f}^{\mathcal{A}} - dA^{\mathcal{B}} \hat{f}^{\mathcal{B}} \\
&= -\rho g^{\mathcal{B}} dt + \sigma_g dW_g^{\mathcal{B}}.
\end{aligned}
$$

Without overconfidence, noises of heterogeneous beliefs can degenerate to 0. Then, differences of groups concerning the volatility $\sigma_f, \sigma_s, \sigma_D$ and the combination of an infinitesimal generator A and mean reversion λ are still considered. However, with $\phi\sigma_f$ overconfidence on fundamental volatility, we can find the fluctuation point of $\phi\sigma_f$ from ρ to σ_g.

Differences in beliefs across agents that will lead to trading. Let us suppose that the optimistic investors A want to bid up prices and happen to hold the whole supply of A and B, for:

$$
p_t^{\mathcal{A}} = \sup_{\tau \geq 0} E_t^{\mathcal{A}}\left[\int_t^{t+\tau} \exp[-r(s-t)] \left[\bar{f} + \exp[-r(s-t)](\hat{f}_t^{\mathcal{A}} - \bar{f})\right] dD_s + \exp(-r\tau)(p_{t+\tau}^{\mathcal{B}} - c)\right],
$$

for the trading cost c : a seller B pays $c \geq 0$ per unit of the asset sold. Scheinkman and Xiong (2003) assumes the following form for the equilibrium price function of group A :

$$
\begin{aligned}
p_t^{\mathcal{A}} &= p^{\mathcal{A}}(A^{\mathcal{A}} \hat{f}_t^{\mathcal{A}}, g_t^{\mathcal{A}}) \\
&= \frac{\bar{f}}{f} + \frac{A^{\mathcal{A}} \hat{f}^{\mathcal{A}} - \bar{f}}{r + \lambda} + q(g_t^{\mathcal{A}}), \\
q(g_t^{\mathcal{A}}) &= \sup_{\tau \geq 0} E_t^{\mathcal{A}}\left[\left(\frac{g_{t+\tau}^{\mathcal{A}}}{r+\lambda}\right) + q(g_t^{\mathcal{B}}) - c\right) \exp(-r\tau)\right].
\end{aligned}
$$

In that, the *bubble* concept relates to the difference between the demand price of the current owner (group A), *i.e.*, group $\mathcal{B}$ in their exposition, and his fundamental valuation along

$$\begin{aligned} b &= q(-k^*) \\ &= q(g^{\mathcal{A}}), \end{aligned}$$

where k^* denotes the minimum difference in opinion—between group $\mathcal{A}$ and group $\mathcal{B}$—that generates a trade.

4.1. Can it be the Speculative Bubble or Overreaction to Surprises

Definition 4.2. *Overconfidence parameter* ϕ increases as a larger ϕ increases, agents attribute to their own forecast of the current level of fundamentals where $0 < \phi < 1$.

It is important to note that $dD_t \approx ds_t^{\mathcal{A}} \approx ds_t^{\mathcal{B}} \approx A\widehat{f^{\mathcal{A}}}dt$ where dD_t denotes the cumulative dividend process , $ds_t^{\mathcal{A}}$ is inferred value with signals of group $\mathcal{A}$ and $A\widehat{f^{\mathcal{A}}}dt$ is mean reverted fundamentals with mean reverted conditional belief and an infinitesimal generator A. Then, conditional mean of beliefs of agents in group $\mathcal{A}$ satisfies

$$\begin{aligned} dA\widehat{f^{\mathcal{A}}} = -\lambda A(\widehat{f^{\mathcal{A}}} - \overline{f})dt + \frac{\phi\sigma_s\sigma_{Af} + \gamma}{\sigma_s^2}(ds^{\mathcal{A}} - A\widehat{f^{\mathcal{A}}}dt) \\ + \frac{\gamma}{\sigma_s^2}(ds^{\mathcal{B}} - A\widehat{f^{\mathcal{A}}}dt) + \frac{\gamma}{\sigma_D^2}(dD - A\widehat{f^{\mathcal{A}}}dt) \end{aligned}$$

where agents in group $\mathcal{A}$ overreact to surprises in $s^{\mathcal{A}}$ because of overconfidence ($\phi > 0$) related to “surprises" with the common volatility of signals σ_s and related Wiener processes are denoted as

$$dW_{\mathcal{A}}^{\mathcal{A}} = \frac{1}{\sigma_s}(ds^{\mathcal{A}} - A\widehat{f^{\mathcal{A}}}dt) \tag{4}$$

$$dW_{\mathcal{B}}^{\mathcal{A}} = \frac{1}{\sigma_s}(ds^{\mathcal{B}} - A\widehat{f^{\mathcal{A}}}dt) \tag{5}$$

and

$$dW_{\mathcal{D}}^{\mathcal{A}} = \frac{1}{\sigma_{\mathcal{D}}}(dD - A\widehat{f^{\mathcal{A}}}dt). \tag{6}$$

Similar representation of equations (4), (5), (6) can be obtained for group $\mathcal{B}$ by simply replacing $A\widehat{f^{\mathcal{A}}}$ by $A\widehat{f^{\mathcal{B}}}$.

4.2. Overconfidence ϕ and an infinitesimal generator A as a speculative bubble crush index

Definition 4.3. *Speculative Bubble Crush Index A* : The infinitesimal generator A as the Speculative Bubble Crush Index parameter is newly modeled from (Scheinkman and Xiong (2003). Regardless of A, agents attribute to their own forecast of the current level of fundamentals as a larger overconfidence ϕ increases (stationary solution agreed by Rogers and Williams 1987; sec. 6.9, Lipster and Shiryaev 1977; theorem 12.7).

$$\gamma \equiv \frac{\sqrt{(A\lambda + \phi\frac{\sigma_f}{\sigma_s})^2 + (1-\phi^2)(2\frac{\sigma_f^2}{\sigma_s^2} + \frac{\sigma_f^2}{\sigma_D^2})} - (A\lambda + \phi\frac{\sigma_f}{\sigma_s})}{\frac{1}{\sigma_D^2} + \frac{2}{\sigma_s^2}}$$

Lemma 4.4. *Stationary variance γ decreases with ϕ.*

Proof. Letting

$$\theta(\phi) = (A\lambda + \phi\sigma_f/\sigma_s),$$
$$\Psi(\phi) = (1-\phi^2)[(2\sigma_f^2/\sigma_s^2) + \sigma_f^2/\sigma_D^2],$$

one derives:

$$\begin{aligned}\frac{d\gamma}{d\phi} &= \frac{[(2\theta(\phi)\theta'(\phi) + \theta'(\phi)]/[2\sqrt{\theta(\phi)^2 + \Psi(\phi)}] - \theta(\phi)}{1/\sigma_D^2 + 2/\sigma_s^2} \\ &= \frac{[\theta(\phi)/\sqrt{\theta(\phi)^2 + \Psi(\phi)} - 1]\theta'(\phi) + \Psi'(\phi)/2\sqrt{\theta(\phi)^2}}{1/\sigma_D^2 + 2/\sigma_s^2} \\ &\leq 0.\end{aligned}$$

□

This is the same result as Lemma 1 in Scheinkman and Xiong (2003). Further, we can extend to the origin of speculative bubble crush.

5. Origin of Speculative Bubble Burst and Unique Trading Point

5.1. Upper Boundary of Speculative Bubble Burst with Overconfidence

Lemma 5.1. *If A (= $\theta'(\lambda)$) is infinitesimal and B (= $\phi\sigma_f/\sigma_s$) is significantly larger than A, the stability Variance γ does not increase as mean reversion λ increases and:*

$$\frac{d\gamma}{d\lambda} = \frac{A}{(A\lambda + B)} - A < 0.$$

Hence, $B\ (= \phi\sigma_f/\sigma_s) > A\lambda$.

Proof. Letting

$$\theta(\lambda) = A\lambda + \phi\sigma_f/\sigma_s,$$
$$\gamma \equiv \sqrt{\theta(\lambda)^2 + \sim} - \theta(\lambda)/(\frac{1}{\sigma_D^2} + \frac{2}{\sigma_s^2}),$$

it is derived that:

$$\begin{aligned}\frac{d\gamma}{d\lambda} &= \Big[\frac{1/2}{\sqrt{[\theta(\lambda)^2]}}2\theta'(\lambda) - \theta'(\lambda)\Big] \Big/ \Big(\frac{1}{\sigma_D^2} + \frac{2}{\sigma_s^2}\Big) \\ &= \Big[\frac{\theta'(\lambda)}{\sqrt{\theta(\lambda)^2}} - \theta'(\lambda)\Big] \Big/ \Big(\frac{1}{\sigma_D^2} + \frac{2}{\sigma_s^2}\Big) \\ &= \theta'(\lambda)\Big(\frac{1}{\theta(\lambda)} - 1\Big) \Big/ \Big(\frac{1}{\sigma_D^2} + \frac{2}{\sigma_s^2}\Big)\end{aligned}$$

Omitting the constant terms $1/\sigma_D^2 + 2/\sigma_s^2$, the sign of $d\gamma/d\lambda$ is the one of

$$A\Big(\frac{1}{\theta(\lambda)} - 1\Big).$$

Remark that

$$\begin{aligned}\theta'(\lambda) &= A \\ &= A\Big(\frac{1}{[A\lambda + (\phi\sigma_f/\sigma_s)]]} - 1\Big).\end{aligned}$$

If A is a nonnegative real number, then

$$\frac{A}{A\lambda + B} - A < 0.$$

If A is infinitesimal, after the mean reversion,

$$A\lambda > A.$$

If $B > A\lambda$, then

$$\frac{A}{2A\lambda} - A < 0,$$

hence $1/A\lambda < 2$ and

$$A/(A\lambda + B) - A < 0.$$

□

Even though the overconfidence parameter ϕ makes constant volatility of fundamentals allocated to the signal of each agent σ_f smaller by σ_f/σ_s, if $\phi\sigma_f/\sigma_s$ is bigger than the mean reverted infinitesimal generator $A\lambda$, stability is not kept by mean reversion process.

5.2. Origin of Speculative Bubble Burst

Theorem 5.2. *(Origin of Speculative Bubble Crush)*
Mean reversion λ doesn't work to maintain the stability variance γ of random process, the infinitesimal generator A is positive oscillation of fundamental value A to maintain the Brownian motion affects mean reversion λ, then threshold of fundamental oscillation $A\lambda$ is captured by the noise movement between $B(=\phi\sigma_f/\sigma_s)$ and $-(\sigma_f/\sigma_s)$, hence

$$-\sigma_f/\sigma_s < A\lambda < B(=\phi\sigma_f/\sigma_s). \tag{7}$$

Proof.

If A is nonnegative and less than 1, then $\dfrac{d\gamma}{d\lambda} = \dfrac{A}{(A\lambda + B)} - A < 0$
It's same with $1 < A\lambda + B$ and $1 - \phi\sigma_f/\sigma_s < A\lambda$
$For \sigma_f > 0, \sigma_s > 0, 0 < (1-\phi)\sigma_f + \sigma_s,$
$0 < \sigma_f - \phi\sigma_f + \sigma_s, -\sigma_f < \sigma_s - \phi\sigma_f, -\dfrac{\sigma_f}{\sigma_s} < \dfrac{\sigma_s}{\sigma_s} - \phi\dfrac{\sigma_f}{\sigma_s}$
$-\sigma_f/\sigma_s < 1 - \phi\sigma_f/\sigma_s < A\lambda$

□

The Breaking point of the forward process can be defined when the speculative bubble crushes , the forward process cannot be completed and the backward induction process is required. If the infinitesimal generator "A" is operated with mean reversion λ, forward process perturbing from the final position which has "bubbles" cannot be processed within the threshold $-\sigma_f/\sigma_s < A\lambda < B(=\phi\sigma_f/\sigma_s)$ in $Af = lim_{t\downarrow 0}(1/t)(P_t f - f)$ where P_t is semi-group. Forward process can be obtained by perturbing the final position, $P_t A f$ is the limit of

$$P_t(1/\varepsilon)(P_\varepsilon f - f)) \text{ as } \varepsilon \to 0.$$

Otherwise, backward induction procedure can be obtained by perturbing the initial position:

$$\partial P_t f = AP_t f = lim_{\varepsilon\to 0}(1/\varepsilon)(P_\varepsilon - I)P_t f.$$

Origin of Speculative Bubble Crush is spotted when dividend noise $\sigma_D dZ_t^D$ is affected by the decrease of $Af_t dt$ in the equation (3).

5.3. Individual Trade using properties of American Option, Smooth Pasting condition. Unique "Trading Point" k^*

Existence For each trading cost $c \geq 0$, there exists a unique trading point k^* where $c(r+\lambda)$ is the expected present value of cost and $h(\cdot)$ is the solution of Ito's lemma condition that solves

$$[k^* - c(r+\lambda)]\,[h'(k^*) + h'(-k^*)] - h(k^*) + h(-k^*) = 0.$$

Continuity of h' If $c = 0$, then $k^* = 0$.

Optimal Exercise Point If $c > 0$, $k^* > c(r+\lambda)$. (Wilmott, 2006)

Depending upon above properties of American option, optimal stopping should be an equilibrium of optimal value function q executed if $g^0 > k^*$, wait for the first if the resale option g^0 is bigger than the trading point k^*. Before pursuing the equilibrium setting, by symmetry of heterogeneous beliefs, it's clear to look the (Scheinkman and Xiong (2003) as followings.

[Bubbles] (Scheinkman and Xiong (2003)

$$q(-k^*) = h(-k^*)/(r+\lambda)\,[h'(k^*) + h'(-k^*)]$$

[Resale option] (Scheinkman and Xiong (2003) for $x < k^*$

$$q(x) = (q(-k^*)/h(-k^*)) \times h(x)$$

for $x \geq k^*$ as below,

$$x/(r+\lambda) + q(-k^*) \times h(-x)/h(-k^*) - c$$

Dana and Le van (2014) pervades an individually rational efficient allocation and an equilibrium setting under two conditions of (1) no arbitrage prices and (2) no unbounded utility arbitrage. It is intended to broaden our viewpoints of understanding the non-negative expectations with respect to risk adjusted probabilities. In connection with this issue, I wish to stretch newly to the argument of individual and collective absence of arbitrage with overconfidence.

6. The unique equilibrium allocation with inertia at the speculative bubble crush

The price system is the l-tuple $p = (p_1, ..., p_h, ..., p_l)$. The value of an action a (arrow, 1959) relative to the price system p is $\sum_{h=1}^{l} p_h a_h$ where a_h is accumulation factor.

A feasible allocation $(x^1, ..., x^I)$ induced from $\sum_i x^i \leq \sum_i w^i$ where i are finitely many agents indexed by $i = 1, ..., I$ and a non-zero price vector $p \in \mathbb{R}^S_+$ are *an equilibrium with inertia* (Rigotti and Shannon, 2005) where S are possible states of nature indexed by $s = 1, ..., S$ if

for all i,

$$x \succ^i x^i \Longrightarrow p \cdot x > p \cdot w^i \tag{8}$$

and endowment $w^i \in R^S_+$ $w^i \in \mathbb{R}^S_+$ is started at date 1 assumed that consumption doesn't occur at date 0 and consumption balance left after trading Arrow securities and

for all i,

$$p \cdot x^i = p \cdot w^i,$$

and for each i, either

$$x^i = w^i,$$

or

$$E_{\pi^i}\left[u^i(x^i)\right] \geq E_{\pi^i}\left[u^i(w^i)\right].$$

where $E_{\pi^\cdot}$ denotes the expected value with respect to the probability distribution π, each $\pi^i \in \Pi^i$, $\exists$ a closed, convex set Π^i and $u(x)$ denotes the vector $u(x_1), ..., u(x_S)$.

Inertia drives an equilibrium in trading only at which is the status quo for all traders so T is the set of agents who are involved in trading in this equilibrium,

$$T = \{i : x^i \neq w^i\}.$$

The number l of commodities is a given positive integer. An action a of an agent is a point of R^l, the commodity space. A price system p is a point of R^l. The value of an action a relative to a price system p is the inner product $p \cdot a$. The equilibrium allocation $(w^1, ..., w^I)$ is supported by p as below,

$$p \cdot a \cdot x^i = p \cdot a \cdot w^i,$$

where a is accumulation factor.
In case of $x < k^*$ where k^* is a trading point,

$$p \cdot \psi(-k, x) \cdot x^i = p \cdot \psi(-k, x) \cdot w^i.$$

In case of $x \geq k^*$,

$$p \cdot \psi(r, \lambda, -k, x) \cdot x^i = p \cdot \psi(r, \lambda, -k, x) \cdot w^i.$$

Hence, speculative bubble crushes at $B(= \phi\sigma_f/\sigma_s)/A > \lambda > -(\sigma_f/\sigma_s)/A$.

Here by, the standard simplex denoted as $\triangle$ is the smallest convex set containing the given vertices in R^S_+. Moreover, given a set $\triangle$, $rint\triangle$ is the relative interior of $\triangle$.

Theorem 6.1. *We assumed that u^i is strictly concave and a closed, convex set of the probability distribution $\Pi^i \subset rint\triangle$. If $(w^1, ..., w^I)$ is an equilibrium allocation, then there is the unique equilibrium allocation with inertia at the speculative bubble crush when $x < k^*$.*

Proof. Two cases are to be considered in their turn.

1. For $x < k^*$, given the resale option function $\psi(-k, x)$, there exists a unique equilibrium allocation:

$$\begin{aligned}
&p \cdot \psi(-k, x) \cdot x^i > p \cdot \psi(-k, x) \cdot w^i,\\
&x^i + (1/n)w^i \succ^i w^i,\\
&p \cdot \psi(-k, x) \left[x^i + (1/n)w^i\right] \succ^i p \cdot \psi(-k, x) \cdot w^i,
\end{aligned}$$

   that implies

$$p \cdot \psi(-k, x) \cdot x^i > p \cdot \psi(-k, x) \cdot w^i$$

   Agents are restricted to the traders involved in trading. Hence, $\{x^i \neq w^i \neq \phi\}$. For example, given a set X, rint(X) denotes the relative interior of X. But

$$\Pi^i \subset \mathrm{rint}(\triangle) = \{\pi \in \triangle : \pi_s > 0, \forall s\}.$$

   For a subset C of $\triangle$ and $C \subset \triangle$, one obtains

$$\mathrm{rint}(C) = \{q \in C : \text{ a neighborhood } V \text{ of } q \text{ in } \mathbb{R}^s$$

$$\text{such that } V \cap \mathrm{rint}(\triangle) \subset C\}.$$

   Now suppose that

$$p \cdot \psi(-k, x) \cdot x^i = p \cdot \psi(-k, x) \cdot w^i.$$

   Fix $\alpha \in (0, 1)$. Then:

$$\begin{aligned}
&p \cdot \psi(-k, x) \cdot w^i\\
&= \alpha p \cdot \psi(-k, x) \cdot w^i - \alpha p \cdot \psi(-k, x) \cdot w^i + p \cdot \psi(-k, x) \cdot w^i\\
&= \alpha p \cdot \psi(-k, x) \cdot w^i + (1 - \alpha)p \cdot \psi(-k, x) \cdot w^i\\
&= \alpha p \cdot \psi(-k, x) \cdot x^i + (1 - \alpha)p \cdot \psi(-k, x) \cdot w^i\\
&= p \cdot \left[(\alpha\psi(-k, x)x^i + (1 - \alpha)\psi(-k, x)w^i\right].
\end{aligned}$$

Whence:

$$E_{\pi^i}\left[u^i(x^i)\right] \geq E_{\pi^i}\left[u^i(w^i)\right],$$

that in turn implies, from the definition of an equilibrium with inertia:

$$E_{\pi^i}\left[u^i(\alpha x^i + (1-\alpha)w^i\right] > E_{\pi^i}\left[w^i\right],\ \forall \pi^i \in \Pi^i,$$

that implies, from the strict concavity of u^i,

$$\begin{aligned} & E_{\pi^i}\left[u^i(\alpha\psi(-k,x)x^i + (1-\alpha)\psi(-k,x)w^i\right] \\ & = E_{\pi^i}\left[\psi(-k,x)w^i\right], \end{aligned}$$

a contradiction.

2. For $x \geq k^*$, given the resale function

$$p \cdot \psi(r,\lambda,-k,x) \cdot x^i = p \cdot \psi(r,\lambda,-k,x) \cdot w^i,$$

$\psi(r,\lambda,-k,x)$ cannot get value at the origin of speculative bubble crush $B(= \phi\sigma_f\sigma_s) > A\lambda > -\sigma_f/\sigma_s$, that completes the statement.

□

7. References

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